

Fuzzy Gravitons From Uncertain Spacetime

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The recently proposed remarkable mechanism explaining “stringy exclusion principle” on an Anti de Sitter space is shown to be another beautiful manifestation of spacetime uncertainty principle in string theory as well as in M theory. Put in another way, once it is realized that the graviton of a given angular momentum is represented by a spherical brane, we deduce the maximal angular momentum directly from either the relation $\Delta t \Delta x^2 > l_p^3$ in M theory or $\Delta t \Delta x > \alpha'$ in string theory. We also show that the result of hep-th/0003075 is similar to results on D2-branes in $SU(2)$ WZW model. Using the dual D2-brane representation of a membrane, we obtain the quantization condition for the size of the membrane.

Mar. 2000

In addition to the by now well-known holographic principle, the spacetime uncertainty principle, first put forward in [1,2] and verified also in the D-brane dynamics [3], is another important principle underlying the yet mysterious grand framework of string/M theory. This was later generalized to M theory in [4]. There are many manifestations of the uncertainty relations [4,5,6]. It is therefore useful to explore as much as possible the physical consequences of this principle. It has been suspected for some time by the present author that the so-called stringy exclusion principle [7] is actually a consequence of the spacetime uncertainty principle. With a remarkable mechanism proposed in a recent paper [8], we will be able to show that indeed this is the case. For the purpose of uncovering the underlying structure of string/M theory, it is a good thing to reduce the number of principles.

It is already pointed out in [8] that with increase of the angular momentum of a massless graviton on S^n , the size of the graviton increases and eventually reaches the size of S^n . And the authors of [8] emphasize rightfully that this is manifestation of space noncommutativity on S^n . Whatever that noncommutativity is, we feel it is worthwhile to point out that this phenomenon fits very nicely with spacetime uncertainty relations which must hold regardless what the background is. We will work with $AdS_7 \times S^4$, $AdS_4 \times S^7$, $AdS_5 \times S^5$ in order. In the end, to lend a support to the mechanism of [8], we show that the result of [8] is similar to results on D-branes in a WZW model of $SU(2)$. To be more accurate, we will use the dual representation of a membrane in M theory as a D2-brane to rederive the result of [8]. In addition, we derive the quantization condition for the size of the membrane.

The spacetime uncertainty relation is

$$\Delta t \Delta x > \alpha'. \quad (1)$$

And the version in M theory reads [4]

$$\Delta t \Delta x \Delta y > l_p^3, \quad (2)$$

where l_p is the Planck length in M theory. The above relation asserts that in M theory any physical process will necessarily involves space uncertainty in two orthogonal directions.

The mechanism of [8] is based on Myers' recent observation that a D0-brane bound state in a constant field $F^{(4)} = dC^{(3)}$ is polarized to become a spherical membrane [9]. In the case of $AdS_7 \times S^4$, a spherical membrane moves on a hemi-sphere parametrized by a

angle ϕ and the membrane size $r \in (0, R)$. As shown in [8], the metric on the round sphere S^4 can be written as

$$ds^2 = \frac{R^2}{R^2 - r^2} dr^2 + (R^2 - r^2) d\phi^2 + r^2 d\Omega_2^2, \quad (3)$$

where the two sphere metric $d\Omega_2^2$ is chosen to coincide with that on the spherical membrane. To see that (r, ϕ) parametrize a round hemi-sphere with radius R , introduce a new angle associated to r through $r = R \sin \psi$, the above metric reads

$$ds^2 = R^2 (d\psi^2 + \cos^2 \psi d\phi^2) + R^2 \sin^2 \psi d\Omega_2^2. \quad (4)$$

Since the range of ψ is $(0, \pi/2)$, clearly the metric on the hemi-sphere parametrized by (ψ, ϕ) is the round metric with radius R . At the north pole $\psi = \pi/2$, the other two sphere parameterizing the membrane has the maximal size R , and at the boundary of the hemi-sphere $\psi = 0$, the size of the membrane vanishes. This is the reason why the collection of all coordinates parameterizes S^4 , not $D_2 \times S^2$. The form (4) will be used later to interpret the membrane as a D2-brane.

The authors of [8] show that if the membrane is allowed to move along ϕ with a fixed size $r = R \sin \psi$, the maximal allowed angular momentum N is achieved when $r = R$. The mechanism for this to happen is similar to a dipole moving on a sphere S^2 with a magnetic field: There is a field strength $F^{(4)}$ on S^4 , the membrane is electrically charged with respect to this magnetic field. Now if one half of the membrane has the orientation such that it is positively charged, then the other half of the membrane has an opposite orientation thus is negatively charged, so the whole membrane is equivalent to a dipole.

Now a spherical membrane with angular momentum n in ϕ direction is an almost BPS state with energy n/R . According to Heisenberg uncertainty relation, the uncertainty in time is given by $\Delta t = R/n$. This graviton has size r in two directions, so

$$\Delta t \Delta x^2 = \frac{R}{n} r^2 \leq \frac{R^3}{n}, \quad (5)$$

since the maximal size of this graviton is R . The spacetime uncertainty relation (2) implies then

$$\frac{R^3}{n} \geq l_p^3, \quad (6)$$

or

$$n \leq \frac{R^3}{l_p^3}. \quad (7)$$

Now $R = l_p(\pi N)^{1/3}$ [10], we thus have

$$n \leq N. \quad (8)$$

We have dropped a factor π since in the uncertainty relation (2) we can not take a numerical factor seriously. We see that the stringy exclusion relation $n \leq N$ is a direct consequence of M theory spacetime uncertainty relation and the fact that in the background $F^{(4)}$, a graviton becomes a round membrane.

According to [9], the size of the membrane r is quantized. If the analysis of [9] in the flat spacetime is directly applicable here, then we would conclude $r \sim n$. We will later give an explanation of this fact using D2-brane representation.

We have thus far ignored the movement of the graviton in the AdS part. To justify our use of spacetime uncertainty relation, we need to show that the movement in AdS does not change the kinematics drastically. Let z be the AdS radial coordinate. We now show that if the graviton starts at z_0 with zero velocity in this direction, its acceleration toward the center of AdS is suppressed by a factor $1/R$. The relevant metric on AdS is

$$ds^2 = \frac{z^2}{R^2} dt^2 - \frac{R^2}{z^2} dz^2. \quad (9)$$

The graviton looks like a massive particle in AdS. So its action is

$$S = -m \int \left(\frac{z^2}{R^2} - \frac{R^2}{z^2} \dot{z}^2 \right)^{1/2} dt. \quad (10)$$

Energy conservation implies

$$(1 - (R/z)^4 \dot{z}^2)^{1/2} = z/z_0.$$

Thus the graviton will move toward the center of AdS. Denote the proper time $d\tau = (z/R)dt$ and the proper radial coordinate $d\rho = R d \ln z$, the acceleration with respect to τ can be easily calculated using the above solution

$$\frac{d^2 \rho}{d\tau^2} = -(z/z_0)^2 \frac{1}{R}. \quad (11)$$

It is seen that this acceleration becomes smaller and smaller as the graviton moves toward $z = 0$.

The case of $AdS_4 \times S^7$ is quite similar. Here one postulates that a graviton is polarized to become a M5-brane. One parameterizes S^7 in a similar way as (3), namely one simply replaces $r^2 d\Omega_2^2$ in (3) by $r^2 d\Omega_5^2$, the metric on the spherical M5 brane. Again for a M5-brane with angular momentum n , the uncertainty in time is $\Delta t \sim R/n$. We can not directly use the uncertainty relation (2) in this case, since the M5-brane has extension in 5 spatial direction. A natural replacement of (2) seems to be $\Delta t \Delta x^5 > l_p^6$. Using this and $\Delta t \Delta x^5 = R r^5 / n \leq R^6 / n$ we find $n < (R/l_p)^6$. Since $R \sim l_p N^{1/6}$ [10], we deduce $n < N$, again the right answer.

It remains to make the new relation $\Delta t \Delta x^5 > l_p^6$ compatible with (2), since if one naively uses the data for the spherical M5-brane, one will find (2) violated. One possible way out of this paradox is to imagine that a five dimensional object consists of a stack of lower dimensional objects, and the effective extension in two out of five spatial directions becomes much larger than it appears. One simple example of this picture is a string behaving like a random walk, its actual size is much larger than what it appears. We believe it is one of future major challenges to formulate a precise mathematical framework to incorporate all these features of uncertainty relations. Curiously, the new relation used for M5-branes is similar to the one valid in the world-volume theory [11]. If there is any connection between the version we used in spacetime physics and the world-volume version, we suspect that this connection is related to our remark on the microscopic structure of M5-branes in terms of more fundamental degrees of freedom.

Turning to the case $AdS_5 \times S^5$. Now the spherical brane is a D3-brane. There are 3 spatial directions involved, again we can not directly apply (1). Note that D3-brane is invariant under S-duality. The appropriate relation respecting S-duality is $\Delta t \Delta x^3 > l_p^4$, where l_p is the Planck length in 10 dimensions, and is $l_p^4 = g_s \alpha'^2$. The spacetime uncertainty relation applied to a moving spherical D3-brane results in $n < R^4 / (g_s \alpha'^2)$. Now $R^4 \sim N g_s \alpha'^2$, again we find $n < N$.

One may ask the question that what happens if there is no corresponding flux on sphere S^n . For this hypothetical background, there would be no dipole mechanism, and thus no restriction on the maximal angular momentum. Therefore the spacetime uncertainty relation is violated. The answer to this question is that such a background is not a consistent solution, and spacetime uncertainty relation ought to hold for a consistent background only. Indeed the original proposal on spacetime uncertainty relation in [4] is based on observations made in a few consistent backgrounds including AdS spaces.

We now show that the result on the giant gravitons in case of $AdS_7 \times S^4$ is similar to recent results on D2-branes in $SU(2)$ WZW model [12]. It is shown that there are $k + 1$ distinct D2-branes on the group manifold S^3 if the level is k . There is a puzzle about this result. There is no S^2 of minimal area in S^3 . This puzzle is recently resolved in [13]. It is argued there that the stabilization of a D2-brane is due to turning on a F field flux on D2-brane. Let us briefly recall a few details. In the large k limit, the metric on S^3 is

$$ds^2 = k\alpha' (d\psi^2 + \sin^2 \psi d\Omega_2^2). \quad (12)$$

To have a 2D CFT, there must be a B field, and it can be written in a certain gauge as

$$B = k\alpha' (\psi - \frac{\sin 2\psi}{2}) d\Omega_2, \quad (13)$$

where $d\Omega_2$ is the volume form on the unit S^2 . In order to stabilize a D2-brane wrapped on S^2 with a constant ψ , one needs to switch on a world-volume F flux:

$$F = -\frac{n}{2} d\Omega_2. \quad (14)$$

n is an integer, since the flux must be quantized. Now the world-volume DBI action is extremized if

$$\psi = \frac{n\pi}{k}. \quad (15)$$

D2-branes are interpreted as M2-branes transverse to the eleventh circle in M theory [14]. The quantization condition on flux F as in (14) is simply the longitudinal momentum quantization, as F is dualized to $\partial_t X_{11}$. The B field (13) is simply the $C^{(3)}$ field in M theory. Thus a D2-brane is interpreted as a membrane moving in the 4 manifold $S^3 \times S^1$ with the angular momentum n along S^1 . Now it is clear that the situation is similar to what is discussed in [9], and to a membrane moving in S^4 . The stabilization of the D2-brane is just as the stabilization of a collection of n D0-branes moving in a constant $F^{(4)}$ field as a spherical membrane.

The maximal angular momentum of a membrane on $S^3 \times S^1$ is k , and at this point the membrane shrinks to a point. This is quite different from the case S^4 , where when the maximal angular momentum is achieved, the membrane has its maximal size. To make this analogy work better, we need to work with the metric (4). We re-interpret S^4 as the tensor product of a hemi-sphere S^3 with S^1 . The former is parametrized by $(\psi, d\Omega_2)$, and the latter by ϕ . Now S^3 has a round metric with radius R , and S^1 has a

ψ -dependent radius $R \cos \psi$, as can be seen from (4). The circle shrinks to zero at the equator of S^3 : $\psi = \pi/2$. If we interpret S^1 as the M theory circle, then the string coupling constant vanishes here. If we take the value R of the maximal radius of ϕ as the canonical eleventh radius, then we have $\alpha' = l_p^3/R$. Re-interpreted in string theory, the radius of S^3 is $R^2 = (R^2/\alpha')\alpha' = N\pi\alpha'$. Due to the existence of the nontrivial dilaton, the string metric on the half S^3 is

$$ds^2 = -\cos \psi dt^2 + N\pi\alpha' \cos \psi (d\psi^2 + \sin^2 \psi d\Omega_2^2), \quad (16)$$

with the dilaton field

$$e^{\frac{2\phi}{3}} = \cos \psi. \quad (17)$$

Note the overall coefficient $N\pi\alpha'$ in (16) is quite different from $k\alpha'$ as in (12).

The $C^{(3)}$ field reads

$$C^{(3)} = \frac{N}{4\pi} \sin^3 \psi d\phi \wedge d\Omega_2. \quad (18)$$

Since the relation between the B field and $C^{(3)}$ is $B_{\mu\nu} = 4\pi^2\alpha' C_{11\mu\nu}$, there is

$$B = N\pi\alpha' \sin^3 \psi d\Omega_2. \quad (19)$$

To check that we have the right B field, note that the flux $\int dB/(2\pi\alpha')$ on the half S^3 is $2\pi N$, the correct quantization condition.

Consider a D2-brane in the half S^3 with a fixed ψ . Again let a constant F flux be switched on as in (14). The DBI action is

$$\begin{aligned} S &= -T_2 \int e^{-\phi} \sqrt{-\det(G + B + 2\pi\alpha' F)} \\ &= -\frac{1}{R} \int dt \cos^{-1} \psi (N^2 \sin^4 \psi - 2nN \sin^3 \psi + n^2)^{1/2}, \end{aligned} \quad (20)$$

where we used $T_2 R = 1/(4\pi^2\alpha')$. This action is extremized if

$$\left(\frac{n}{N} - \sin \psi\right)(\sin^3 \psi - 2 \sin \psi + \frac{n}{N}) = 0. \quad (21)$$

We find the solution

$$\sin \psi = \frac{n}{N}. \quad (22)$$

Indeed the allowed maximal angular momentum is $n = N$. We have dropped a factor $\cos^{-2} \psi$ in (21). When $\sin \psi = 1$, this is a singular factor. But it is easy to see that $\sin \psi = 1$ is a zero of order 3 in (21).

The static action (20) is just $-\int dt V$, and V is the static potential. Thus $E = V$. Substituting $\sin \psi = n/N$ into E we find $E = n/R$. This is just the statement that the energy of the D2-brane is equal to its longitudinal momentum in direction ϕ , a consistency check that indeed we have the right graviton. It can be checked, though a little tediously, that at $\sin \psi = n/N$, the second derivative of $E(\psi)$ is positive, thus these solutions are stable:

$$\frac{d^2 E(\psi)}{d\psi^2} = \frac{1}{R} \frac{n^3}{N^2 - n^2}. \quad (23)$$

This quantity diverges at $n = N$, indicating that the maximal membrane is very stable. We have ignored other solutions coming from $\sin^3 \psi - 2 \sin \psi + n/N = 0$, since they are unstable solutions.

Back to the M theory metric, the radius of the membrane is $r = R \sin \psi = nR/N$. When the membrane has the maximal angular momentum, it has the maximal size. This agrees with the result of [8]. This is not a surprise, since we know that the D2-brane picture is dual to the membrane picture, although the procedure of executing calculations is quite different here. Although we are considering a curved manifold S^4 , the quantization of r seems to be the same as for a membrane moving in the flat spacetime with a constant $F^{(4)}$ field.

Much more can be done with our D2-brane approach, for instance we can analyze the spectrum in the world-volume theory, and the noncommutativity of S^2 . So this approach seems to have more advantages than the original approach in [8].

Without much effort, the above discussion can be generalized to $AdS_4 \times S^7$ and $AdS_5 \times S^5$.

We end this paper with a discussion of what kind of fuzzy spheres we obtain from AdS. It is proposed in [15,16] that the spheres are actually quantum deformed spheres with deformation parameter $q = \exp(i2\pi/N)$. This proposal is based on the phenomenological observation that the representations of the associated quantum group terminate. The mechanism of [8] seems to suggest a different picture. Focus on the case $AdS_7 \times S^4$. As we already explained in the beginning, S^4 can be viewed as the tensor product of a hemisphere S^2 with a constant radius and a sphere S^2 with variable radius r . The latter is wrapped by a membrane with a quantized radius $r = nR/N$. Viewed in M theory, this is a fuzzy sphere whose fuzziness is determined by n . In terms of the D2-brane picture, the effective flux $\mathcal{F} = B/(2\pi\alpha') + F \sim n(1 - n^2/N^2)$. Thus the world-volume theory on this D2-brane is a noncommutative field theory. The membrane behaves like a dipole on

the hemi-sphere with a magnetic field. So this hemi-sphere is another fuzzy sphere whose fuzziness is determined by N . We seem to obtain a fuzzy S^4 as the tensor product of a fuzzy hemi-sphere and a fuzzy sphere. It remains to construct a precise mathematical framework for this fuzzy S^4 .

Acknowledgments. This work was supported by a grant of NSC and by a “Hundred People Project” grant of Academia Sinica. I thank P.M. Ho and Y.S. Wu for discussions.

Note added: We are informed that refs.[15],[16] predate [8] in emphasizing noncommutativity of spheres. Also, [17] discusses general features of space uncertainties which may be relevant to situation discussed here.

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